doi:10.1006/jabr.2000.8421, available online at http://www.idealibrary.com on IDEAL®

Schur Subalgebras and an Application to the Symmetric Group

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Communicated by Gordon James

Received February 15, 2000

Let K be an infinite field of prime characteristic p and let $d \leq r$ be positive integers of the same parity satisfying a certain congruence condition. We prove that the Schur algebra $S(2, d)$ is isomorphic to a subalgebra of the form $eS(2, r)e$, where e is a certain idempotent of $S(2, r)$. Translating this result via Ringel duality to the symmetric groups Σ_d and Σ_r, we obtain lattice isomorphisms between Specht modules, between Young modules, and between permutation modules. Here modules labelled by the partitions $(r - k, k)$ correspond to modules labelled by $(d - k, k)$. This provides a representation theoretical interpretation for part of the fractal structures observed for the decomposition numbers of the symmetric groups corresponding to two-part partitions.

1. INTRODUCTION

Let K be an infinite field of prime characteristic p and let n, r be positive integers. It has long been known that representations of the symmetric and general linear groups are closely related. This relationship was first discovered by Schur in his doctoral thesis [23] in 1901 and has since been refined and simplified. In particular, in 1980, Green [11] clarified this relationship by introducing certain algebras, the Schur algebras. The Schur algebra $S(n, r)$ over K is a finite-dimensional, associative algebra whose module category is equivalent to the category of r-homogeneous polynomial representations of the general linear group $GL_n(K)$. Let Σ_r be the symmetric group on r symbols. Then the Schur algebra $S(n, r)$ can be defined as the endomorphism ring $\mathrm{End}_{K\Sigma_r}(E^{\otimes r})$, where E is an n-dimensional vector space over K and where the symmetric group acts on $E^{\otimes r}$ by place permutations.

0021-8693/00 $35.00

For $d \in \mathbb{N}$, define the function $h : \mathbb{N} \to \mathbb{N}_0$ by $h(d) := \max\{e \mid p^e \leq d\}$. Let $d \leq r$ be natural numbers of the same parity and such that $r \equiv d \bmod p^{h(d)+1}$. The first main result of this paper is that the Schur algebra $S(2, d)$ is isomorphic to a subalgebra of $S(2, r)$ which is of the form $eS(2, r)e$, for a certain idempotent $e \in S(2, r)$ (see Theorems 5.1 and 5.2). Here we use the results on Cartan matrices in Henke [12] and we find a suitable translation functor (see Jantzen [19, II.7]). Moreover we obtain applications to representations of symmetric groups, by using general results on quasi-hereditary algebras. Let I_r be the kernel of the action of $K\Sigma_r$ on the tensor space $E^{\otimes r}$. We show that $K\Sigma_r/I_r$ is Morita equivalent to a factor algebra of $K\Sigma_d/I_d$, and this equivalence identifies Specht modules, simple modules, Young modules labelled by $(r-k, k)$ with those labelled by $(d-k, k)$ (with a minor modification for $p = 2$ and r even). In particular there exists a strong lattice isomorphism between $S^{(d-k,k)}$ and $S^{(r-k,k)}$ (see Theorem 6.4).

2. BACKGROUND

2.1. *The General Linear Groups*

Let $G = GL_n(K)$ and T be a maximal torus of G. We denote by $X(T) = \mathbb{Z}^n$ the set of weights of G, by $X^+(T) = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in X(T) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$ the set of dominant weights of G, and we define the set of p^m-restricted weights of G by $X_m^+(T) = \{(\lambda_1, \ldots, \lambda_n) \in X^+(T) \mid 0 \leq \lambda_i - \lambda_{i+1} < p^m \text{ for } 1 \leq i \leq n-1\}$.

For $\lambda \in X^+(T)$, let k_λ be the one-dimensional T-module on which T acts with weight λ. Define $\nabla(\lambda)$ as the induced module $\nabla(\lambda) = \operatorname{Ind}_T^G(k_\lambda)$. The module $\nabla(\lambda)$ is indecomposable and its socle is simple, denoted by $L(\lambda)$. The set of modules $L(\lambda)$ for all $\lambda \in X^+(T)$ gives a set of representatives of rational simple G-modules. Among the rational G-modules $\nabla(\lambda)$, for $\lambda \in X^+(T)$, the polynomial ones are precisely those modules $\nabla(\lambda)$, where $\lambda \in \mathbb{N}^n \subseteq X^+(T)$. Similarly, the simple polynomial modules are precisely those $L(\lambda)$, where $\lambda \in \mathbb{N}^n \subseteq X^+(T)$. Any polynomial module is a direct sum of homogeneous polynomial modules, in particular, an indecomposable polynomial G-module is homogeneous of some degree r, where $r \in \mathbb{N}$.

2.2. *The Schur Algebra*

The category of finite-dimensional modules of $S(n, r)$ is equivalent to the category of homogeneous polynomial G-modules of degree r, where the simple $S(n, r)$-modules are precisely given by the simple polynomial G-modules of degree r. They are parametrized by $\Lambda^+(n, r)$, the set of partitions of r with at most n-parts and are denoted again by $L(\lambda)$

(for $\lambda \in \Lambda^+(n,r)$). Every G-module $\nabla(\lambda)$, where $\lambda \in \Lambda^+(n,r)$, is homogeneous of degree r and is identified with an $S(n,r)$-module (same notation). The so-called Weyl module or standard module $\Delta(\lambda)$ is defined to be the contravariant dual of $\nabla(\lambda)$ (also called dual Weyl module or costandard module). The Weyl modules and the dual Weyl modules of $S(n,r)$ belong each to a unique block of $S(n,r)$. The decomposition matrix D of a Schur algebra $S(n,r)$ is the matrix recording the composition factors of (dual) Weyl modules; the Cartan matrix C gives the composition factors of the indecomposable projective modules. Since $C = D^t D$ where D^t denotes the transpose of D, the structure of the Cartan matrix is determined by the structure of the decomposition matrix D. For details on these results on Schur algebras, see the monograph by Green [11].

We consider henceforth $n = 2$. Every two-part partition $\lambda = (\lambda_1, \lambda_2)$ of r is uniquely determined by $\lambda_1 - \lambda_2$. We therefore equivalently denote the simple module $L(\lambda)$ by $L(\lambda_1 - \lambda_2)$. A similar notation applies to the (dual) Weyl modules. We say λ has size $\lambda_1 - \lambda_2$. Then the natural order of integers orders partitions by size. Note that the difference $\lambda_1 - \lambda_2$ is smaller than or equal to $\mu_1 - \mu_2$ if and only if the partition (λ_1, λ_2) is dominated by the partition (μ_1, μ_2).

2.3. *The Symmetric Group*

The Specht modules of the group algebra $K\Sigma_r$ are parametrized by partitions of r and we denote the Specht module corresponding to the partition λ by S^λ. The simple $K\Sigma_r$-modules can be parametrized by p-regular partitions of r, and the simple module corresponding to μ is denoted by D^μ; here D^μ is the unique simple quotient of S^μ. The decomposition matrix of the group algebra $K\Sigma_r$ is the matrix recording the multiplicities $[S^\lambda : D^\mu]$ of the simple $K\Sigma_r$-module D^μ as a composition factor of the Specht module S^λ. For details on these results see the monograph by James [17].

The triangle of binomial coefficients, considered modulo a prime p, is a classical example for a so-called Sierpinski gasket. This fractal structure is geometrically built up from a series of triangles; for example, for $p = 2$, three smaller triangles make the next bigger type of triangle, etc. (see Fig. 1). For the Sierpinski gasket see Stewart [24]. The decomposition numbers $[S^\lambda : D^\mu]$ of $K\Sigma_r$ corresponding to two-part partitions λ, μ have been determined by James [15, 16]. The decomposition matrices can be viewed as a deformed Sierpinski gasket. This result, so far, has been considered as a combinatorial result. We expect that many of the arithmetic properties which give rise to the Sierpinski gasket can be interpreted in the language of representation theory. In this paper we give an example for this: the decomposition numbers for two-part partitions form the following self-repeating pattern. Let $d \leq r$ be natural numbers with the same parity

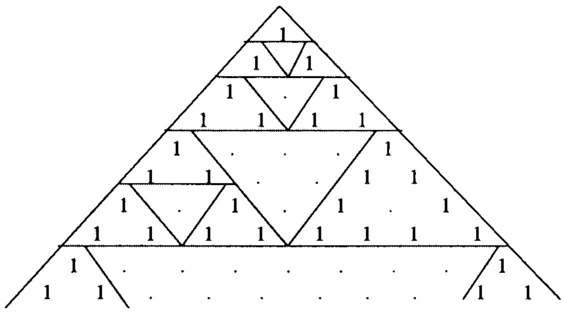

FIG. 1. The triangle of binomial coefficients modulo 2.

and such that $r \equiv d \bmod p^{h(d)+1}$. Let j and k be non-negative integers such that $(d-k, k)$ and $(d-j, j)$ are partitions. We obtain from James's calculation of the decomposition numbers the equation

$$[S^{(r-k,\,k)} : D^{(r-j,\,j)}] = [S^{(d-k,\,k)} : D^{(d-j,\,j)}].$$

This combinatorial result has an interpretation in representation theory: using the embedding of the Schur algebra $S(2, d)$ into $S(2, r)$, we prove in particular (see Theorem 6.4) that there exists a strong lattice isomorphism between $S^{(r-k,\,k)}$ and $S^{(d-k,\,k)}$.

3. ON THE DECOMPOSITION MATRIX OF $S(2, r)$

Suppose the p-adic decomposition of $n \in \mathbb{N}_0$ is given by $n = \sum_{i\in\mathbb{N}_0} n_i p^i$ with $0 \le n_i < p$. We define $[n]_i := n_i$ for all $i \in \mathbb{N}_0$. For a real number x we denote the largest integer less than or equal to x by $[x]$. We define the function $f = f_p : \mathbb{N}^2 \to \mathbb{N}$ by

$$f(x, y) = \prod_{i\in\mathbb{N}_0} \binom{p-1-[x]_i}{p-1-[y]_i},$$

where x, y are non-negative integers. By Henke [12], we have the following explicit description of the decomposition numbers of $S(2, r)$:

PROPOSITION 3.1. *Let $s, t \in \mathbb{N}_0$ have the same parity. Then the simple module $L(s)$ is a composition factor of $\Delta(t)$ if, and only if, $f(s, (s+t)/2) = 1$.*

It is well known that the multiplicity of a simple module as a composition factor of some Weyl module for $n = 2$ is at most one. As an application of Proposition 3.1 we obtain:

LEMMA 3.1. *Let $d \le r$ be natural numbers of the same parity and such that $r \equiv d \bmod p^{h(d)+1}$ and let $0 \le s, t \le d$. Then*

$$[\Delta(t) : L(s)] = [\Delta(t + r - d) : L(s + r - d)].$$

Proof. We have $f(x, y) = 1$ if and only if $[x]_i = [y]_i$ or $[y_i] = p - 1$ for all $i \in \mathbb{N}_0$. Hence $[\Delta(t) : L(s)] = 1$ if and only if $[s]_i = [(s + t)/2]_i$ or $[(s + t)/2]_i = p - 1$. By hypothesis $r - d = a \cdot p^{h(d)+1}$ for some $a \in \mathbb{N}_0$ and hence we obtain for all $i \in \mathbb{N}_0$: $[s]_i = [s + r - d]_i$ and $[(s + t)/2]_i = [(s + t)/2 + r - d]_i$. Hence $[\Delta(t) : L(s)] = 1$ if and only if $[\Delta(t + r - d) : L(s + r - d)] = 1$. ∎

Let $C(2, r)$ be the Cartan matrix of $S(2, r)$. We label the rows (columns) of $C(2, r)$ increasingly, going from top (left) to bottom (right). Then we have seen:

COROLLARY 3.1. *Let $d, r \in \mathbb{N}$ be such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Then the Cartan matrix $C(2, d)$ is a submatrix in the right-hand bottom corner of $C(2, r)$.*

If the characteristic $p = 2$, we can improve this result. Since it will not be used in this paper, we state the result without providing a proof. (The proof is an easy exercise or can be found in Henke [14].)

PROPOSITION 3.2. *Let d be an even natural number. Then the Cartan matrix $C(2, d)$ is a submatrix in the right-hand bottom corner of $C(2, r)$ if and only if $r \equiv d$ modulo $2^{h(d)}$.*

4. COMPARING $GL_n(K)$-HOMOMORPHISMS

In this section we provide some technical background for the proof of our main results in Section 5. In particular we are interested in the following: Let m and n be non-negative integers with $n \geq 1$. Let K be an infinite and perfect field of prime characteristic p and let $G = GL_n(K)$ be the general linear group over K. Let V and W be rational G-modules, denote by V^* the (usual) dual of V, and let det be the determinant module. Furthermore, let $F : G \to G, (g_{ij}) \mapsto ({g_{ij}}^p)$ be the Frobenius twist. Then the following isomorphism (of vector spaces) holds,

$$\begin{aligned} \mathrm{Hom}_G(V, W) &\cong W \otimes_K V^* \\ &\cong W \otimes_K \det{}^m \otimes_K (\det{}^*)^m \otimes_K V^* \\ &\cong \mathrm{Hom}_G(V \otimes_K \det{}^m, W \otimes_K \det{}^m), \end{aligned}$$

for $m \in \mathbb{Z}$; if $V = W$ this isomorphism is a ring isomorphism. Since the determinant module is one-dimensional and since tensoring of a module by a one-dimensional module induces a (strong) submodule-lattice isomorphism (see Subsection 5.1), this result is not particularly surprising. However, it allows us to compare the set of G-homomorphisms of modules of different homogeneous degree. In this section we generalize this.

4.1. *Comparing Homomorphisms*

Let V and W be rational G-modules whose dominant weights are p^m-restricted. We are interested in finding simple rational G-modules L such that

$$\operatorname{Hom}_G(V, W) \cong \operatorname{Hom}_G(V \otimes_K L^{F^m}, W \otimes_K L^{F^m}). \tag{1}$$

The vector spaces $\operatorname{Hom}_G(V \otimes_K L^{F^m}, W \otimes_K L^{F^m})$ and $\operatorname{Hom}_G(V, W \otimes_K (L \otimes_K L^*)^{F^m})$ are isomorphic. The functor $- \otimes_K (L \otimes_K L^*)^{F^m}$ is a functor on the category of rational G-modules which induces a map

$$t = t_L : \operatorname{Hom}_G(V, W) \to \operatorname{Hom}_G(V, W \otimes_K (L \otimes_K L^*)^{F^m}). \tag{2}$$

This map t_L is defined by sending $\varphi \in \operatorname{Hom}_G(V, W)$ to the map $t(\varphi)$, which is given by $t(\varphi)(v) = \varphi(v) \otimes_K \mathrm{id}$. For a non-zero map φ, we choose an element $v \in V$ such that $\varphi(v) \neq 0$. Then $t(\varphi)(v) = \varphi(v) \otimes_K \mathrm{id} \neq 0$ and hence the map $t = t_L$ is injective. In order to establish Eq. (1) we are therefore interested in finding simple modules L such that the map t_L is also surjective. There exists a short exact sequence of G-modules (see Jantzen [19, I.2.7(4)])

$$0 \to K \to (L \otimes_K L^*)^{F^m} \to X \to 0 \tag{3}$$

for some $X = X(L, m)$. Applying to this short exact sequence the left exact functor $\operatorname{Hom}_G(V, W \otimes_K -)$, we obtain the left exact sequence

$$0 \to \operatorname{Hom}_G(V, W) \to \operatorname{Hom}_G(V, W \otimes_K (L \otimes_K L^*)^{F^m}) \to \operatorname{Hom}_G(V, W \otimes_K X).$$

We adjust the map t_L with respect to Eq. (3). By the above (slightly modified) considerations we get the following lemma:

LEMMA 4.1. *Let V and W be rational G-modules with p^m-restricted dominant weights, let L be a simple rational G-module and let t_L and X be defined as above. Furthermore, suppose the vector space* $\operatorname{Hom}_G(V, W \otimes_K X) = 0$. *Then the map t_L is an isomorphism.*

So far the situation has been completely general. In the following, we specialise our considerations to simple modules L which are equal to a costandard module. As a consequence, we can apply the Littlewood–Richardson rule (see James and Kerber [18, Corollary 2.8.14]). We then apply Lemma 4.1.

LEMMA 4.2. *Let V and W be rational G-modules and suppose V has a ∇-filtration. Then the module $W \otimes_K V$ is filtered by the modules $L(\mu) \otimes_K \nabla(\lambda)$, where $L(\mu)$ runs through all composition factors of W and $\nabla(\lambda)$ runs through all ∇-modules occurring in a ∇-filtration of V.*

Proof. This follows by the exactness of the functors $- \otimes_K V$ and $L(\mu)\otimes_K$ and induction on the filtration length. ∎

LEMMA 4.3. *Let M be a module filtered by modules M_i for $1 \leq i \leq k$. Then* $\operatorname{soc}(M) \subseteq \oplus_{i=1}^k \operatorname{soc}(M_i)$.

Proof. This follows by induction on the filtration length and by the isomorphism theorem: Let $0 \to M_1 \to M \to M_2 \to 0$ be a short exact sequence of modules and suppose that the simple module L lies in $\operatorname{soc}(M)$, but is not a submodule of M_1. Then $L \cong L/(L \cap M_1) \cong (L + M_1)/M_1 \leq M/M_1 \cong M_2$. Hence L lies in the socle of M_2. ∎

PROPOSITION 4.1. *Let $m \in \mathbb{N}$ and let U and W be rational G-modules such that U has a filtration by modules $\nabla(\gamma_i)^{F^m}$ for $1 \leq i \leq k$ and such that the composition factors $L(\alpha)$ of W have p^m-restricted dominant weight. Then* $\operatorname{soc}(W \otimes_K U) \subseteq \oplus_{\alpha, i} L(\alpha) \otimes_K L(\gamma_i)^{F^m}$, *where $L(\alpha)$ runs through the composition factors of W.*

Proof. Combining Lemma 4.2 and Lemma 4.3, the socle of $W \otimes_K U$ is contained in $\sum_{\alpha, i} \operatorname{soc}(L(\alpha) \otimes_K \nabla(\gamma_i)^{F^m})$. The module $L(\alpha) \otimes_K \nabla(\gamma_i)^{F^m}$ is filtered by modules $L(\alpha) \otimes_K L(\beta)^{F^m}$, where $L(\beta)$ is a composition factor of $\nabla(\gamma_i)$. We obtain

$$\begin{aligned}
&\operatorname{Hom}_G(L(\alpha) \otimes_K L(\beta)^{F^m}, L(\alpha) \otimes_K \nabla(\gamma_i)^{F^m}) \\
&\quad \cong ((L(\alpha)^* \otimes_K L(\alpha) \otimes_K (L(\beta)^* \otimes_K \nabla(\gamma_i))^{F^m})^{G_m})^G \\
&\quad \cong ((L(\alpha)^* \otimes_K L(\alpha))^{G_m} \otimes_K (L(\beta)^* \otimes_K \nabla(\gamma_i))^{F^m})^G,
\end{aligned}$$

where G_m denotes the mth Frobenius kernel. Since the module $L(\alpha)_{|G_m}$ is simple, we obtain $(L(\alpha)^* \otimes_K L(\alpha))^{G_m} \cong K$. In the above chain of isomorphisms we are therefore left with

$$\begin{aligned}
(L(\beta)^* \otimes_K \nabla(\gamma_i)^{F^m})^G &\cong \operatorname{Hom}_G(L(\beta)^{F^m}, \nabla(\gamma_i)^{F^m}) \\
&\cong \begin{cases} 0 & \text{if } \beta \neq \gamma_i, \\ K & \text{if } \beta = \gamma_i. \end{cases}
\end{aligned}$$

By Steinberg's tensor product theorem, the modules $L(\alpha) \otimes_K L(\gamma_i)^{F^m}$ are simple. Hence $\operatorname{soc}(W \otimes_K U) \subseteq \oplus_{\alpha, i} \operatorname{soc}(L(\alpha) \otimes_K L(\gamma_i)^{F^m}) = \oplus_{\alpha, i} L(\alpha) \otimes_K L(\gamma_i)^{F^m}$, where $L(\alpha)$ runs through the composition factors of W. ∎

THEOREM 4.1. *Let V, W be rational G-modules such that all of their dominant weights are p^m-restricted, some $m \in \mathbb{N}$. Let $L = L(\lambda)$ be a simple rational G-module with $L(\lambda) = \nabla(\lambda)$. Then*

$$\operatorname{Hom}_G(V, W) \cong \operatorname{Hom}_G(V \otimes_K L^{F^m}, W \otimes_K L^{F^m})$$

is an isomorphism of vector spaces and if $V = W$ this is a ring isomorphism.

Proof. Let the notation be as above. We show that $\mathrm{Hom}_G(V, W \otimes_K X) = 0$ and then apply Lemma 4.1. Without further reference we will make frequent use of Lemma 4.2.

(1) Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an element in $X^+(T)$ such that $L(\lambda) = \nabla(\lambda)$. Since simple polynomial G-modules are (contravariant) self-dual, we obtain $L(\lambda) = \nabla(\lambda) = \Delta(\lambda)$ and hence $\nabla(\lambda)^* = \Delta(\lambda)^*$. Then

$$\begin{aligned}\nabla(\lambda)^* &\cong \Delta(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)\\ &\cong \nabla(-\lambda_n, -\lambda_{n-1}, \ldots, -\lambda_1)\\ &\cong \nabla(\mu) \otimes_K \det{}^{-\lambda_1},\end{aligned}$$

where $\mu = (\lambda_1 - \lambda_n, \lambda_1 - \lambda_{n-1}, \ldots, \lambda_1 - \lambda_2, 0)$. Hence

$$\begin{aligned}L(\lambda) \otimes_K L(\lambda)^* &\cong \nabla(\lambda) \otimes_K \nabla(\lambda)^*\\ &\cong \nabla(\lambda) \otimes_K \nabla(\mu) \otimes_K \det{}^{-\lambda_1}.\end{aligned}$$

Since $\nabla(\lambda) \otimes_K \nabla(\mu)$ has a ∇-filtration, the latter module is filtered by modules of the form $\nabla(\gamma) \otimes_K \det^{-\lambda_1}$. By the Littlewood–Richardson rule (see James and Kerber [18, Corollary 2.8.14] and Fulton and Harris [9, Lecture 6]), the multiplicity of $K \cong \nabla(\lambda_1^n) \otimes_K \det^{-\lambda_1}$ in this filtration is equal to one. Hence for $L = L(\lambda)$, the module $X \cong (L(\lambda) \otimes_K L(\lambda)^*)^{F^m}/K$ (see Eq. (3)) is filtered by $\nabla(\gamma_i)^{F^m} \otimes_K \det^{-\lambda_1 p^m}$ with $1 \leq i \leq k$ and some $k \in \mathbb{N}_0$, where in this filtration the module $\nabla(\lambda_1^n)^{F^m} \otimes_K \det^{-\lambda_1 p^m}$ does not occur.

(2) Define the module $U := X \otimes_K \det^{\lambda_1 p^m}$. Then the module $W \otimes_K U$ is filtered by modules $L(\alpha) \otimes_K \nabla(\gamma_i)^{F^m}$, where $L(\alpha)$ runs through the simple composition factors of W. By hypothesis the dominant weights α are p^m-restricted. Hence, by Proposition 4.1, the socle of $W \otimes_K U$ contains only composition factors of the form $L(\alpha) \otimes_K L(\gamma_i)^{F^m} = L(\alpha + p^m \gamma_i)$, where $L(\alpha)$ is a composition factor of W. By the first part of the proof, the module $L(\alpha) \otimes_K L(\lambda_1^n)^{F^m}$ does not occur in the socle of $W \otimes_K U$.

We consider next the module $V \otimes_K \det^{\lambda_1 p^m}$. It has composition factors of the form $L(\beta) \otimes_K \det^{\lambda_1 p^m} = L(\beta + p^m(\lambda_1^n))$. Since α and β are p^m-restricted, the equation $\alpha + p^m \gamma_i = \beta + p^m(\lambda_1^n)$ implies that $\gamma_i = (\lambda_1^n)$ and hence $\alpha = \beta$. We conclude that the module $V \otimes_K \det^{\lambda_1 p^m}$ has no common composition factor with $\mathrm{soc}(W \otimes_K U)$. Then

$$\mathrm{Hom}_G(V, W \otimes_K X) \cong \mathrm{Hom}_G(V \otimes_K \det{}^{\lambda_1 p^m}, W \otimes_K U) = 0$$

and hence, by Lemma 4.1, the map $t_{L(\lambda)}$ in Eq. (2) is an isomorphism. ∎

If $n = 2$, Theorem 4.1 will hold even without the hypothesis $L(\lambda) = \nabla(\lambda)$. Suppose $\lambda = (\lambda_1, \lambda_2)$ is any dominant weight whose p-adic decomposition into p-restricted partitions $\lambda(i)$ is given by $\lambda = \sum_{i=0}^{k} \lambda(i)p^i$, some $k \in \mathbb{N}_0$. Then $L(\lambda) = \otimes_{i=0}^{k} L(\lambda(i))^{F^i}$. Since $\lambda(i)$ is p-restricted, $L(\lambda(i)) = \nabla(\lambda(i))$, for $0 \leq i \leq k$. Let V, W be rational G-modules all of whose dominant weights are p^m-restricted. Then, by Theorem 4.1,

$$\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}_G(V \otimes_K L(\lambda(0))^{F^m}, W \otimes_K L(\lambda(0))^{F^m}).$$

The dominant weights of the rational G-modules $V' := V \otimes_K L(\lambda(0))^{F^m}$ and $W' := W \otimes_K L(\lambda(0))^{F^m}$ are p^{m+1}-restricted. Hence we can apply Theorem 4.1 with $L(\lambda(1))$ twisted $(m+1)$-times. By induction on the decomposition length k of the p-adic decomposition of λ and by applying again Steinberg's tensor product theorem we obtain:

COROLLARY 4.1. *Let $n = 2$ and let V, W be rational G-modules such that all of their dominant weights are p^m-restricted. Let $L = L(\lambda)$ be a simple rational G-module. Then* $\mathrm{Hom}_G(V, W) \cong \mathrm{Hom}_G(V \otimes_K L^{F^m}, W \otimes_K L^{F^m})$ *is an isomorphism of vector spaces and if $V = W$ this is a ring isomorphism.*

4.2. *An Application to $GL_2(K)$*

Let A be a finite-dimensional K-algebra. Let P be an indecomposable A-module with head isomorphic to the simple module S. Suppose the projective indecomposable module corresponding to S has the same composition factors with the same multiplicities as P. Then P is isomorphic to the projective indecomposable module corresponding to S.

PROPOSITION 4.2. *Let $m \in \mathbb{N}$. Let $\mu \in X^+(T)$ be such that $L(\mu) = \nabla(\mu)$ or such that μ is a partition into two parts. Let $\lambda \in X_m^+(T)$ and let P be an indecomposable, polynomial G-module with head isomorphic to the simple module $L(\lambda)$. Suppose all composition factors of P have a dominant weight which is p^m-restricted. Then $P \otimes_K L(\mu)^{F^m}$ is an indecomposable module with head isomorphic to $L(\lambda + \mu p^m)$.*

Proof. Let $\beta \in X_m^+(T)$. Then the composition factors of $P \otimes_K L(\mu)^{F^m}$ are of the form $L(\beta) \otimes_K L(\mu)^{F^m}$ and they are simple by Steinberg's tensor product theorem. Hence by Theorem 4.1 or Corollary 4.1

$$\begin{aligned} \mathrm{Hom}_G(P \otimes_K L(\mu)^{F^m}, L(\beta) \otimes_K L(\mu)^{F^m}) &\cong \mathrm{Hom}_G(P, L(\beta)) \\ &\cong \begin{cases} 0 & \text{if } \lambda \neq \beta, \\ K & \text{if } \lambda = \beta. \end{cases} \end{aligned}$$

The head of $P \otimes_K L(\mu)^{F^m}$ is therefore simple and hence the module is indecomposable. ∎

COROLLARY 4.2. *Let $\lambda \in X_m^+(T)$ and let μ and P be given as in the previous proposition. Furthermore, suppose that $P \otimes_K L(\mu)^{F^m}$ and its projective cover have the same dimension. Then $P \otimes_K L(\mu)^{F^m}$ is the projective indecomposable module corresponding to the simple module $L(\lambda + p^m \mu)$.*

The modules in Proposition 4.2 and Corollary 4.2 are indecomposable polynomial modules. Hence they are homogeneous, which means, they are modules for a suitable Schur algebra. This will be used in Section 5.

5. SUBALGEBRAS OF THE SCHUR ALGEBRA $S(2, r)$

5.1. *Submodule Lattices and Endomorphism Rings*

Let A be a finite-dimensional algebra. We denote the submodule lattice of an A-module V by $\mathcal{M}(V) := \{W | W \leq_A V\}$. Let V and W be A-modules. Then a map $f : \mathcal{M}(V) \to \mathcal{M}(W)$ is a *lattice homomorphism* if f preserves inclusion: if $V_1 \subseteq V_2$ then $f(V_1) \subseteq f(V_2)$. If f is bijective, it is called a lattice isomorphism. Let M and N be A-modules and let $\phi : M \to N$ be an A-module homomorphism. An idempotent $e \in A$ induces a functor $F : \text{mod}_A \to \text{mod}_{eAe}$, given by $F(M) = Me$ and $F(\phi) = \phi_{|Me} : Me \to Ne$. As a vector space, Me is isomorphic to $\text{Hom}_A(eA, M)$ (see Benson [1, Lemma 1.3.3]). The following lemma is well known:

LEMMA 5.1. *The functor F induces a surjection from the submodule lattice of the A-module M onto the the submodule lattice of the eAe-module Me. Furthermore, if every composition factor L of M satisfies $Le \neq 0$, then this lattice homomorphism is an isomorphism.*

Since the functor F is exact, the lattice isomorphism induced by F satisfies the following property: if $M_2 \subseteq M_1$, then $F(M_1/M_2) \cong F(M_1)/F(M_2)$. We call this a *strong* lattice isomorphism. For example, an equivalence of module categories induces a strong submodule-lattice isomorphism. The following two lemmas are well known:

LEMMA 5.2. *Suppose every composition factor L of M satisfies the condition $Le \neq 0$. Then the homomorphism $\text{End}_A(M) \to \text{End}_{eAe}(Me)$, given by $\phi \mapsto F(\phi)$, is injective.*

LEMMA 5.3. *Let M and N be A-modules and let e be the projection of $M \oplus N$ with image M and kernel N. Then the algebra $\text{End}_A(M)$ is isomorphic to the algebra $e\text{End}_A(M \oplus N)e$.*

5.2. *Progenerators and Morita Equivalence*

Let K be a field and let A and B be finite-dimensional K-algebras. An A-module X is called a *generator* for mod_A if for all A-modules M there exists a natural number n and an epimorphism $X^n \to M$. A finitely generated projective generator is a *progenerator*. An A-module X_A is a generator for the category mod_A if and only if every indecomposable projective A-module is isomorphic to a direct summand of X. By Morita's theorem (see Benson [1, Theorem 2.2.6]), the algebras A and B are Morita equivalent if and only if there exists a progenerator P_B for B such that $A \cong \mathrm{End}_B(P_B)$.

PROPOSITION 5.1. *Let A and B be Morita equivalent algebras whose simple modules are parametrized by the set Λ. For $\lambda \in \Lambda$ we denote the corresponding simple modules by $L_A(\lambda)$ or $L_B(\lambda)$, respectively. If $\dim_K L_A(\lambda) = \dim_K L_B(\lambda)$ for all $\lambda \in \Lambda$, then the algebras A and B are isomorphic.*

Proof. Suppose A and B are Morita equivalent algebras, where in this equivalence $L_B(\lambda)$ corresponds to $L_A(\lambda)$. This induces an isomorphism of division rings,

$$\mathrm{End}_B(L_B(\lambda)) \cong \mathrm{End}_A(L_A(\lambda)).$$

We define $D(\lambda) := \mathrm{End}_B(L_B(\lambda))$. By Morita's theorem, there exists a progenerator Q_A for A with $\mathrm{End}_A(Q_A) \cong B$. Let $\Omega \in \{A, B\}$. We denote by $P_\Omega(\lambda)$ the projective indecomposable Ω-module corresponding to $L_\Omega(\lambda)$. Let $[\Omega_\Omega : P_\Omega(\lambda)]$ be the multiplicity of $P_\Omega(\lambda)$ in a direct sum decomposition of the regular representation Ω_Ω. Let $n_{\lambda,\Omega} = [\Omega_\Omega : P_\Omega(\lambda)]$. Then the semisimple quotient of Ω corresponding to $L_\Omega(\lambda)$ is isomorphic to the matrix ring $M_{n_{\lambda,\Omega}}(D(\lambda))$. Hence, by Wedderburn's theorem, $n_{\lambda,\Omega}\dim_K(L_\Omega(\lambda)) = \dim_K(M_{n_{\lambda,\Omega}}(D(\lambda))) = n^2_{\lambda,\Omega}\dim_K(D(\lambda))$ and

$$\dim_K(L_\Omega(\lambda)) = n_{\lambda,\Omega}\dim_K(D(\lambda)).$$

We conclude, by using the hypothesis, that $n_{\lambda,A} = n_{\lambda,B}$. Then the multiplicity with which the module $P_A(\lambda)$ occurs in a direct sum decomposition of Q_A is

$$[Q_A : P_A(\lambda)] = [B_B : P_B(\lambda)] = n_{\lambda,B} = n_{\lambda,A} = [A_A : P_A(\lambda)],$$

where the first equality follows by Fitting's lemma. Hence the modules Q_A and A_A are isomorphic and so $B \cong \mathrm{End}_A(Q) \cong \mathrm{End}_A(A) \cong A$. ∎

5.3. *The Main Theorem*

Let K be an infinite and perfect field of prime characteristic p and consider the Schur algebra $S(n,r)$ defined over K. Let F be the Frobenius twist for Schur algebras.

THEOREM 5.1. *Let $d, r \in \mathbb{N}$ be such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Then the Schur algebra*

$S(2,d)$ is Morita equivalent to a centralizer subalgebra of the Schur algebra $S(2,r)$.

Proof. We assume that d and r are both even. The proof for d and r both odd can be carried out in a similar way.

Step 1. For abbreviation let $A := S(2,r)$. The indexing sets

$$I_r = \{0, 2, \dots, r\},$$

$$I_d = \{0, 2, \dots, d\},$$

both consisting of even non-negative integers only, parametrize the simple modules and the projective indecomposable modules of $A = S(2,r)$ and $S(2,d)$, respectively. For $i \in I_r$ and $i \in I_d$ we denote the simple modules by $L_r(i)$ and $L_d(i)$, respectively, and we denote the corresponding projective indecomposable modules by $P_r(i)$ and $P_d(i)$, respectively. We choose a decomposition of 1_A as a sum of primitive orthogonal idempotents ϵ_{ij}, where the labelling is chosen such that $i \in I_r$, $j \in \{1, \dots, n_i\}$ for some $n_i \in \mathbb{N}$ and $\epsilon_{ij}A \cong \epsilon_{kl}A$ if and only if $i = k$. Let $e_i := \epsilon_{i1}$ and let the labelling be such that $e_i A = P_r(i)$ is the projective indecomposable module corresponding to the simple module $L_r(i)$. Define

$$e = \sum_{i \in I_r,\, i \geq r-d} e_i \tag{4}$$

and consider the basic algebra eAe. We denote its simple and projective indecomposable modules by L_{eAe} and P_{eAe}, respectively. The simple and projective indecomposable eAe-modules are parametrized by

$$I_{eAe} := \{i \in I_r \mid i \geq r - d\}$$

and for $i \in I_{eAe}$ they are given by $L_{eAe}(i) = L_r(i)e$ and $P_{eAe}(i) = P_r(i)e$, respectively. Then, by construction, for $i, j \in I_{eAe}$,

$$[P_r(i) : L_r(j)] = [P_{eAe}(i) : L_{eAe}(j)]. \tag{5}$$

Step 2. We define P_d to be the direct sum of all non-isomorphic projective indecomposable $S(2,d)$-modules, where each occurs exactly once. Let

$$m := h(d) + 1.$$

Then, by assumption, there exists a natural number a such that $r = d + ap^m$. Let $i \in I_d$. Since $d < p^m$, each weight λ of $P_d(i)$ is p^m-restricted. Hence every composition factor of $P_d(i)$ is isomorphic to a module $L_d(j)$, where j corresponds to a weight which is p^m-restricted. By Steinberg's tensor product theorem, the module $L_d(j) \otimes_K L_a(a)^{F^m}$ is simple. This has two consequences: First, by Proposition 4.2, the module $P_d(i) \otimes_K L_a(a)^{F^m}$ is indecomposable with head isomorphic to $L_r(i + ap^m) = L_r(r - d + i)$.

Second, by Theorem 4.1,

$$\mathrm{End}_{S(2,d)}(P_d) \cong \mathrm{End}_{S(2,r)}(P_d \otimes_K L_a(a)^{F^m}). \tag{6}$$

Step 3. We saw in the previous step that for every $i \in I_d$ the module $P_d(i) \otimes_K L_a(a)^{F^m}$ is indecomposable with head isomorphic to $L_r(r - d + i)$. Furthermore, by construction, every composition factor of $P_d(i) \otimes_K L_a(a)^{F^m}$ is of the form $L_r(r - d + j)$. Hence, by the definition of e in Eq. (4), for every composition factor L of $P_d(i) \otimes_K L_a(a)^{F^m}$ the module Le is non-zero. By Lemma 5.1 and the remark following it, the module $(P_d(i) \otimes_K L_a(a)^{F^m})e$ is indecomposable with head isomorphic to $L_r(r - d + i)e = L_{eAe}(r - d + i)$. Let $i, j \in I_d$. Using Eq. (5) and Proposition 3.2 we obtain

$$\begin{aligned}
&[P_{eAe}(r-d+i) : L_{eAe}(r-d+j)] \\
&\quad = [P_r(r-d+i) : L_r(r-d+j)] \\
&\quad = [P_d(i) : L_d(j)] \\
&\quad = [P_d(i) \otimes_K L_a(a)^{F^m} : L_d(j) \otimes_K L_a(a)^{F^m}] \\
&\quad = [(P_d(i) \otimes_K L_a(a)^{F^m})e : (L_d(j) \otimes_K L_a(a)^{F^m})e]
\end{aligned}$$

and hence, by Corollary 4.2, the module $(P_d(i) \otimes_K L_a(a)^{F^m})e$ is isomorphic to the projective indecomposable module $P_{eAe}(r - d + i)$. The set $\{P_r(r - d + i)e \mid i \in I_d\}$ is a complete set of representatives of projective indecomposable eAe-modules. By construction each of these modules is a direct summand of $(P_d \otimes_K L_a(a)^{F^m})e$. Hence the module $(P_d \otimes_K L_a(a)^{F^m})e$ is a progenerator for the basic algebra eAe.

Step 4. We finally derive the following inclusion for the basic algebra $S_0(2, d)$ of $S(2, d)$,

$$\begin{aligned}
S_0(2,d) &\cong \mathrm{End}_{S(2,d)}(P_d) \\
&\cong \mathrm{End}_A(P_d \otimes_K L_a(a)^{F^m}) \qquad \text{by Eq. (6),} \\
&\subseteq \mathrm{End}_{eAe}((P_d \otimes_K L_a(a)^{F^m})e) \qquad \text{by Lemma 5.2,} \\
&\cong eAe = eS(2,r)e,
\end{aligned}$$

where the last isomorphism follows by Step 3. Since K is a splitting field, the dimensions of the algebras above are determined by the Cartan matrices. By Corollary 3.1, the dimension of $S_0(2, d)$ and of eAe coincide and the inclusion above is, in fact, an isomorphism. We conclude that the algebra $S(2, d)$ is Morita equivalent to $eAe = eS(2, r)e$ and thus to a subalgebra of $A = S(2, r)$. ∎

5.4. *Collection of Results*

The proof of Theorem 5.1 contains several statements about the structure of Schur algebras for $n = 2$ and about the structure of their modules. We list them in the following:

THEOREM 5.2. *Let $d, r \in \mathbb{N}$ be defined as above and let $e \in S(2, r)$ be chosen as in Eq.* (4). *Let P_d be the direct sum of all non-isomorphic projective indecomposable $S(2, r)$-modules, each occurring exactly once. Let $m = h(d) + 1$, let $r = d + a \cdot p^m$, and let $I_d = \{i \in \mathbb{N}_0 \mid i \leq d, i \text{ even}\}$.*

(a) $\mathrm{End}_{S(2,d)}(P_d)$ *and* $\mathrm{End}_{S(2,r)}(P_d \otimes_K L_a(a)^{F^m})$ *are isomorphic algebras.*

(b) *The module $P_d(i) \otimes_K L_a(a)^{F^m}$ is indecomposable with head isomorphic to $L_r(r - d + i)$. Its composition factors are of the form $L_r(r - d + j)$ with $L_r(r - d + j)e \neq 0$.*

(c) *The module $(P_d(i) \otimes_K L_a(a)^{F^m})e$ is the projective indecomposable module corresponding to $L_{eS(2,r)e}(r - d + i)$. In particular, the module $(P_d \otimes_K L_a(a)^{F^m})e$ is a progenerator for $eS(2, r)e$.*

(d) *For all $i \in I_d$ there exists a strong lattice isomorphism between the submodule lattices of $P_r(r - d + i)e$ and of $P_d(i)$.*

(e) *The module $P_r(r - d + i)$ is the projective cover of $P_d(i) \otimes_K L_a(a)^{F^m}$. If π is the corresponding projection, then* $\ker(\pi)$ *contains all composition factors $L_r(j)$ for $j < r - d$ and no others.*

DEFINITION 5.1. Let A be a finite-dimensional K-algebra, let M be an A-module, and let $\pi : P \to M$ be its projective cover. Then M is *almost projective* if for all $\alpha \in \mathrm{End}_A(P)$ we have $\alpha(\ker(\pi)) \subseteq \ker(\pi)$.

Almost projective modules are studied in Dipper [2, 3] and Schubert [22]; in [22] we are given some equivalent reformulations of the property of a module being almost-projective. As a consequence of Theorem 5.2 we obtain the following corollary:

COROLLARY 5.1. *Let the notation be as above. Then the module $P_d(i) \otimes_K L_a(a)^{F^m}$ is almost projective as a module for $S(2, r)$.*

We conclude this section with an improvement of Theorem 5.1; we show that there exists an idempotent $\bar{e} \in S(2, r)$ such that the algebras $S(2, d)$ and $\bar{e}S(2, r)\bar{e}$ are isomorphic. In order to obtain this result, we need to adjust the multiplicities in our choice of the idempotent e in the proof of Theorem 5.1. Let $\lambda \in \Lambda^+(n, r)$ and let $n_\lambda \in \mathbb{N}_0$ be the multiplicity of the Young module Y^λ in $E^{\otimes r}$. Then

$$S(n, r) = \mathrm{End}_{K\Sigma_r}(\oplus_{\lambda \in \Lambda^+(n,r)} n_\lambda Y^\lambda). \tag{7}$$

By Henke [13], we have the following lemma about Young modules:

LEMMA 5.4. *Let $d, r \in \mathbb{N}$ be such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Let k, s be such that $(d-k, k)$ and $(d-s, s)$ are partitions. Then the Young module $Y^{(r-k,k)}$ is a direct summand of the permutation module $M^{(r-s,s)}$ if and only if the Young module $Y^{(d-k,k)}$ is a direct summand of the permutation module $M^{(d-s,s)}$.*

LEMMA 5.5. *Let $d, r \in \mathbb{N}$ be such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Let k be such that $(d-k, k)$ is a partition. Then $n_{(d-k,k)} \leq n_{(r-k,k)}$.*

Proof. For a partition λ of r we denote by $a(\lambda)$ the cycle type of λ. It is given by $\lambda = (r^{a_r(\lambda)}, \ldots, 1^{a_1(\lambda)})$. We assume that d and r are both even. Let $0 \leq s \leq d/2$. By Grabmeier [10, Satz 8.11], the multiplicity of $M^{(r-s,s)}$ as a direct summand in a direct sum decomposition of $E^{\otimes r}$ is $\binom{n}{a(r-s,s)} = \binom{2}{1,1} = 2$ and hence is smaller than or equal to the multiplicity of $M^{(d-s,s)}$ as a direct summand in a direct sum decomposition of $E^{\otimes d}$. The latter, in turn, is $\binom{2}{a(d-s,s)} = \binom{2}{2} = 1$ if $s \neq d/2$ and is $\binom{2}{a(d-s,s)} = \binom{2}{1,1} = 2$ if $s = d/2$. By Lemma 5.4, we have $n_{(d-k,k)} \leq n_{(r-k,k)}$. If d and r are both odd, the proof can be carried out in a similar way. ∎

COROLLARY 5.2. *Suppose $d, r \in \mathbb{N}$ are such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Then the Schur algebra $S(2, d)$ is isomorphic to a subalgebra of the Schur algebra $S(2, r)$.*

Proof. First, we assume that d and r are both even. By Eq. (7) the Schur algebra $S(2, r)$ is given by

$$S(2, r) = \operatorname{End}_{k\Sigma_r}\Big(\oplus_{k=0}^{r/2} n_{(r-k,k)} Y^{(r-k,k)} \Big),$$

where $n_{(r-k,k)} = \dim L_r(r-k, k)$ for all $0 \leq k \leq r/2$. For $0 \leq k \leq d/2$ we define $m_{(r-k,k)} = \dim L_d(d-k, k)$. By Lemma 5.5, $m_{(r-k,k)} \leq n_{(r-k,k)}$ for all $0 \leq k \leq d/2$. We fix a direct sum decomposition of $E^{\otimes r}$ into Young modules and take $\lambda \in \Lambda^+(2, r)$. Then there exist n_λ copies of Y^λ in this decomposition of $E^{\otimes r}$. By abuse of notation we denote the projection onto any such copy by e_λ. Let $e \in S(2, r)$ be defined as $e = \sum_{k=0}^{d/2} e_{(r-k,k)}$. Note that e can be identified with the idempotent defined in Eq. (4) in the proof of Theorem 5.1. We modify e with respect to multiplicities and define

$$\bar{e} = \sum_{k=0}^{d/2} \sum_{i=1}^{m_{(r-k,k)}} e_{(r-k,k)}. \tag{8}$$

By Lemma 5.3, the algebra $R := \operatorname{End}_{K\Sigma_r}(\oplus_{k=0}^{d/2} m_{(r-k,k)} Y^{(r-k,k)})$ is isomorphic to the algebra $\bar{e}S(2, r)\bar{e}$, which, in turn, is Morita equivalent to the algebra $eS(2, r)e$. Hence, by Theorem 5.1, the algebra R is Morita equivalent to $S(2, d)$. By construction, the simple R-modules are parametrized by $\Lambda = \{(r-k, k) \mid 0 \le k \le d/2\}$ and we denote the simple R-module corresponding to $\lambda \in \Lambda$ by $L_R(\lambda)$. Then

$$\dim L_R(r-k, k) = m_{(r-k,\,k)} = \dim L_d(d-k, k).$$

Hence, by Proposition 5.1, the algebra $S(2, d)$ is isomorphic to the subalgebra R of the Schur algebra $S(2, r)$. If d and r are both odd, the proof can be carried out in a similar way. ∎

Of course, Theorem 5.2 can be adjusted to the situation in Theorem 5.2 and its proof.

6. AN APPLICATION TO THE SYMMETRIC GROUPS

6.1. *Quasi-hereditary Algebras*

Let K be any field and let A be a quasi-hereditary K-algebra with respect to $(\Lambda, \le)$. Without loss of generality we assume that the algebra A is basic. We fix a set of primitive orthogonal idempotents e_λ, where e_λ corresponds to the simple module with the same parameter $\lambda \in \Lambda$. The definitions and results presented in this section can be found, for example, in König [20] or in the appendix A, written by Dlab, of the book by Drozd and Kirichenko [7]. We assume $\Lambda = \{1, \dots, n\}$ with the natural order $1 < \cdots < n$. (By Dlab and Ringel [4] this is no restriction.) The module category mod_A is called a *highest weight* category and the elements of Λ are called *weights*. An A-module V has *highest weight* λ if $L(\lambda)$ is a composition factor of V and for all composition factors $L(\mu)$ of V, we have $\mu \le \lambda$. Once the partial ordering of a quasi-hereditary algebra is fixed, the standard modules are determined uniquely up to isomorphism. For each $\lambda \in \Lambda$ we denote the corresponding standard module by $\Delta(\lambda)$ and the corresponding costandard module by $\nabla(\lambda)$. Let Γ be a subset of Λ and let $e_\Gamma := \sum_{\lambda \in \Gamma} e_\lambda$. The set $\Gamma \subseteq \Lambda$ is *saturated* if $\mu \in \Gamma$ and $\lambda \le \mu$ implies $\lambda \in \Gamma$. The set $\Lambda \backslash \Gamma$ is saturated if and only if $(\Gamma, \le^{op})$ is saturated.

PROPOSITION 6.1. *Let Γ be a subset of Λ such that $\Lambda\backslash\Gamma$ is saturated. Let $e = e_\Gamma$ be defined as above. Then the algebras $\overline{A} = A/AeA$ and eAe are quasi-hereditary.*

For $\Lambda = \{1, \dots, n\}$ with the natural order $1 < \cdots < n$, we define the algebras $\overline{A}_i = A/A\epsilon_{i+1}A$ and $A_i = \epsilon_i A \epsilon_i$ for $1 \le i \le n$, where $\epsilon_i := e_i + \cdots + e_n$ (for $1 \le i \le n$) and $\epsilon_{n+1} = 0$. Then the algebras $\overline{A}_i$ are quasi-hereditary with respect to $\overline{\Lambda}_i := \{1, \dots, i\}$. The ideal $A\epsilon_{i+1}A$ annihilates the A-standard modules and $\Delta_{\overline{A}_i}(j) = \Delta_A(j)$, with $1 \leq j \leq i$. The algebras A_i are quasi-hereditary with respect to $\Lambda_i := \{i, \dots, n\}$ for $j \in \Lambda_i$. The standard module $\Delta_{A_i}(j)$ is given by $\Delta_{A_i}(j) = \Delta_A(j)\epsilon_i$. We remark that similar statements as for standard modules also hold for tilting modules (to be defined below).

6.2. *The Ringel Dual*

An A-module V has a Δ-*filtration* if there exists a sequence of submodules $V = V_1 \supseteq V_2 \supseteq \cdots \supseteq V_k = 0$ of V such that every subquotient is either zero or isomorphic to some $\Delta(\lambda)$ for $\lambda \in \Lambda$. The multiplicity of $\Delta(\lambda)$ in such a filtration of V is independent of the choice of filtration and we denote it by $[V : \Delta(\lambda)]$. The module V is also called a Δ-*good* module. Similar definitions and statements hold in the dual case. Denote by $\mathfrak{F}(\Delta)$ the full subcategory of A-modules consisting of Δ-*good* objects and by $\mathfrak{F}(\nabla)$ the full subcategory of A-modules consisting of ∇-*good* objects.

THEOREM 6.1 (Ringel [21]). *For each $\lambda \in \Lambda$ there is a unique indecomposable module $T(\lambda)$ in the intersection of $\mathfrak{F}(\Delta)$ and $\mathfrak{F}(\nabla)$ with highest weight λ.*

The modules $T(\lambda)$ whose existence is asserted in Theorem 6.1 are called *tilting* modules. The module $T_A = T := \oplus_{\lambda\in\Lambda} T(\lambda)$ or more generally $T_A = T := \oplus_{\lambda\in\Lambda} n_\lambda T(\lambda)$, with $n_\lambda \in \mathbb{N}$, is called a *full* or *characteristic* tilting module. Let $\mathfrak{T} := \mathfrak{F}(\Delta) \cap \mathfrak{F}(\nabla)$ be the full subcategory of A-modules whose objects have both a Δ-good and a ∇-good filtration. The category $\mathfrak{T}$ is closed under direct sums and under tensor products (as are already the categories $\mathfrak{F}(\nabla)$ and $\mathfrak{F}(\Delta)$).

DEFINITION 6.1. For a quasi-hereditary algebra A with full tilting module T_A, $A' := \mathrm{End}_A(T_A)$ is called a *Ringel dual* of A.

So by definition, the Ringel dual of A is, up to Morita equivalence, uniquely determined. Moreover:

THEOREM 6.2 (Ringel [21]). *A Ringel dual A' of a quasi-hereditary algebra A is again a quasi-hereditary algebra on the same indexing set, but with reversed order.*

We consider the functor $\mathrm{Hom}_A(T, -)$, which maps A-modules to A'-modules. Then the projective indecomposable modules of A' are given by $P'(\lambda) = P_{A'}(\lambda) = \mathrm{Hom}_A(T, T(\lambda))$, the standard modules by $\Delta'(\lambda) = \Delta_{A'}(\lambda) = \mathrm{Hom}_A(T, \nabla(\lambda))$ and the tilting modules by $T'(\lambda) = T_{A'}(\lambda) = \mathrm{Hom}_A(T, I(\lambda))$, where $I(\lambda)$ is the injective hull of the module $L(\lambda)$. Hence, in particular, A-modules with a ∇-filtration are mapped to A'-modules with a Δ-filtration. Indeed, the functor $\mathrm{Hom}_A(T, -)$

induces an equivalence of $\mathfrak{F}_A(\nabla)$ and $\mathfrak{F}_{A'}(\Delta)$. For all these statements see, for example, Donkin [6, Appendix A4].

We iterate the process of taking the Ringel dual. Let A' be a Ringel dual of the algebra A. The algebra A'' obtained by taking a Ringel dual of A' is Morita equivalent to the algebra A, such that standard and costandard modules are identified again and the labelling is preserved.

For all $\lambda \in \Lambda$ we take the primitive orthogonal idempotent e'_λ to be the projection from T onto $T(\lambda)$. The following can be deduced form Ringel [21], using the characterization of quasi-hereditary algebras in terms of tilting modules.

PROPOSITION 6.2. *Let Γ be a subset of Λ such that $\Lambda \backslash \Gamma$ is saturated. Let $\overline{A}$ be defined as in Proposition* 6.1 *and let A' be the endomorphism ring of a full, multiplicity-free tilting module of A and let $e' = e'_\Gamma$.*

(1) *The algebra $\overline{A}$ has Ringel dual isomorphic to $e'A'e'$.*

(2) *The algebra eAe is Morita equivalent to the Ringel dual of $A'/A'e'A'$.*

6.3. *An Application to the Symmetric Group*

We exploit a connection between the Ringel dual $S(n, r)'$ of the Schur algebra $S(n, r)$ and the group algebra $K\Sigma_r$ of the symmetric group on r symbols. The r-fold tensor product space $E^{\otimes r}$ of the n-dimensional K-vector space E is a left $S(n, r)$-module and a right $K\Sigma_r$-module and these actions commute. By Donkin [5], $T(\lambda)$ occurs as a direct summand of $E^{\otimes r}$ if and only if λ is p-regular. We define $C := \oplus_\mu T(\mu)$, where the sum is taken over all $\mu \in \Lambda^+(n, r)$ which are not p-regular. Let $T := E^{\otimes r} \oplus C$. This is a full tilting module and we take $S(n, r)' = \mathrm{End}_{S(n,r)}(T)$. Let $\pi : T \to T$ be the projection of T with kernel C, let $\rho_n : K\Sigma_r \to \mathrm{End}(E^{\otimes r})$ be the representation corresponding to the right $K\Sigma_r$-module $E^{\otimes r}$, and let $I_r := \ker(\rho_n)$. Then by Erdmann [8, Proposition 4.3], the centralizer algebra $\pi S(n, r)'\pi$ is Morita equivalent to the quotient $K\Sigma_r/I_r$. By the definition of $\rho = \rho_n : K\Sigma_r \to \mathrm{End}(E^{\otimes r})$, I_r is equal to the annihilator of $E^{\otimes r}$. In particular, I_r annihilates every Young module, every Specht module, and every simple module. We therefore study these modules without restriction as modules over $K\Sigma_r/I_r$.

THEOREM 6.3 (Erdmann [8, Proposition 4.3]). *The Morita equivalence between $K\Sigma_r/I_r$ and $\pi S(n, r)'\pi$ has the following properties*:

(1) *Let $\lambda, \mu \in \Lambda^+(n, r)$. The Young module Y^λ is identified with $T'(\lambda)\pi$ and the Specht module S^λ with $\Delta'(\lambda)\pi$. Hence, in particular, we have*

$$[T'(\lambda)\pi : \Delta'(\mu)\pi] = [Y^\lambda : S^\mu].$$

(2) *Let* $\lambda \in \Lambda^+(n,r)$ *be* p*-regular and assume that all partitions* μ *in the same block as* λ *and with* $\mu > \lambda$ *are* p*-regular. Then there exists a strong submodule-lattice isomorphism between* $\Delta'(\lambda)\pi$ *and* $\Delta'(\lambda)$ *and between* $T'(\lambda)\pi$ *and* $T'(\lambda)$.

Note that in (2) the idempotent π is the identity and $S(n,r)'$ is isomorphic to $K\Sigma_r/I_r$. We specialise to $n=2$. For $p=2$ and r even the partition $(r/2, r/2)$ is the largest weight with respect to $\leq^{op}$. Hence the simple module $L'(r/2, r/2)$ does not occur in $T'(\lambda)$ and $\Delta'(\lambda)$. We obtain:

COROLLARY 6.1. *Let* $\lambda, \mu \in \Lambda^+(2,r)$ *and assume that in case* $p=2$ *and* r *is even the partition* $\lambda \neq (r/2, r/2)$. *Then there exist strong submodule-lattice isomorphisms between* $\Delta'(\lambda)\pi$ *and* $\Delta'(\lambda)$ *and between* $T'(\lambda)\pi$ *and* $T'(\lambda)$. *In particular we obtain*

$$[Y^\lambda : S^\mu] = [T'(\lambda) : \Delta'(\mu)] = [\Delta(\mu) : L(\lambda)] = [\nabla(\mu) : L(\lambda)].$$

THEOREM 6.4. *Let* $d, r \in \mathbb{N}$ *be such that* $r \equiv d$ *modulo* $p^{h(d)+1}$ *and such that* d *and* r, *with* $d \leq r$, *have the same parity. Furthermore, let* k *be a nonnegative integer such that* $(d-k, k)$ *is a partition. In case* $p=2$ *and* r *is even, we assume that* $k \neq d/2$. *Then there exist strong submodule-lattice isomorphisms between* $Y^{(r-k,k)}$ *and* $Y^{(d-k,k)}$, *between* $S^{(r-k,k)}$ *and* $S^{(d-k,k)}$, *and between* $M^{(r-k,k)}$ *and* $M^{(d-k,k)}$.

Proof. A partition $(\lambda_1, \lambda_2) \in \Lambda^+(2,r)$ is uniquely determined by $r = \lambda_1 + \lambda_2$ and the difference $\lambda_1 - \lambda_2$. We therefore equivalently use the parametrizing set $\{\lambda_1 - \lambda_2 | (\lambda_1, \lambda_2) \in \Lambda^+(2,r)\}$ and a sub-index r to denote modules which are parametrized by $\Lambda^+(2,r)$. We adopt a similar notation for modules parametrized by $\Lambda^+(2,d)$.

Define $i := d-2k$ and identify $T'(d-k,k)$ with $T'_d(i)$. Similarly, using that $r-2k = (r-d)+i$, we identify $T'(r-k,k)$ with $T'_r(r-d+i)$.

(1) We apply Theorem 6.3 to both $K\Sigma_r$ and $K\Sigma_d$ and therefore distinguish the projection π in Theorem 6.3 by writing π_d and π_r, respectively. Then $Y^{(d-k,k)}$ is identified with $T'_d(i)\pi_d$ and $Y^{(r-k,k)}$ with $T'_r(r-d+i)\pi_r$. By Corollary 6.1 we can also identify $Y^{(d-k,k)}$ with $T'_d(i)$ and $Y^{(r-k,k)}$ with $T'_r(r-d+i)$.

(2) By the proof of Theorem 5.1, the algebra $S(2,d)$ is isomorphic to $\bar{e}S(2,r)\bar{e}$, where $\bar{e} \in S(2,d)$ is defined as in Eq. (8). Then

$$S(2,d)' \cong S(2,r)'/S(2,r)'\bar{e}'S(2,r)',$$

where $\bar{e}'$ is defined as in Proposition 6.2. Hence $S(2,r)'\bar{e}'S(2,r)'$ operates trivially on $T'_r(j)$ for $j \geq r-d$ and $T'_d(i)$ is identified with $T'_r(r-d+i)$.

Combining (1) and (2) and by Proposition 5.1, we obtain a strong submodule-lattice isomorphism between $Y^{(d-k,k)}$ and $Y^{(r-k,k)}$. The proof of the result for Specht modules and permutation modules can be carried out similarly. ∎

COROLLARY 6.2. *Let $d, r \in \mathbb{N}$ be such that $r \equiv d$ modulo $p^{h(d)+1}$ and such that d and r, with $d \leq r$, have the same parity. Furthermore, if $p = 2$ assume that r is odd. Then $K\Sigma_d/I_d$ is Morita equivalent to a quasi-hereditary quotient of $K\Sigma_r/I_r$. Let $k \in \mathbb{N}_0$ be such that $(d-k, k)$ is a partition. Under the equivalence of the corresponding module categories the module $D^{(d-k,k)}$ is mapped to $D^{(r-k,k)}$, $S^{(d-k,k)}$ is mapped to $S^{(r-k,k)}$, $Y^{(d-k,k)}$ is mapped to $Y^{(r-k,k)}$, and $M^{(d-k,k)}$ is mapped to $M^{(r-k,k)}$.*

We conclude with several remarks: While comparing blocks of the symmetric groups, quite a few results on blocks with the same weight and different core are known. The reader should note that the blocks of the modules $S^{(d-k,k)}$ and $S^{(r-k,k)}$ above have the same core but different weights. The arguments given in this section hold more generally and we expect similar results as for $n = 2$ for natural numbers n with $n < p$. All of the above results strongly rely on the calculation of the Cartan numbers; it would be interesting to have a proof of the above results without using Cartan numbers.

ACKNOWLEDGMENTS

This paper is part of my D. Phil. thesis. I gratefully acknowledge financial support from the Bischöflichen Studienstiftung Cusanuswerk and the Engineering and Physical Sciences Research Council. I wholeheartedly thank my supervisor K. Erdmann for countless discussions on Schur algebras and Ringel duality.

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